Question 1

1. $u_0^* = \frac{q_0}{q_0+\lambda_0}$ and $u_1^* = \frac{q_1}{q_1+\lambda_1}$. Since the unemployment rate is increasing in the inflow rate to unemployment and decreasing in the exit rate out of unemployment, $q_1 > q_0$ and $\lambda_1 < \lambda_0$ both imply that $u_1^* > u_0^*$.

2. The change in unemployment between time t and $t+dt$ is given by

$$\begin{aligned} u(t+dt) - u(t) &= (1-u(t))q_1 dt - u(t)\lambda_1 dt \Leftrightarrow \\ \frac{u(t+dt)-u(t)}{dt} &= (1-u(t))q_1 - u(t)\lambda_1 \\ \dot{u}_t = \lim_{dt\to 0} \frac{u(t+dt)-u(t)}{dt} &= q_1 - (\lambda_1 + q_1)u(t) \end{aligned}$$

3. Plugging into the solution $x(t) = Ce^{-at} + \frac{b}{a}$, we have that

$$u(t) = Ce^{-(\lambda_1+q_1)t} + \frac{q_1}{q_1+\lambda_1} \tag{1}$$

   Evaluating this in $t = 0$ and noticing that $u_1^* = \frac{q_1}{q_1+\lambda_1}$

$$\begin{aligned} u_0^* &= Ce^0 + u_1^* \Leftrightarrow \\ C &= -(u_1^* - u_0^*) \end{aligned}$$

   Inserting this in equation (1) yields

$$\begin{aligned} u(t) &= -(u_1^* - u_0^*)e^{-(\lambda_1+q_1)t} + u_1^* \Leftrightarrow \\ &= u_0^* + (u_1^* - u_0^*)\left[1 - e^{-(\lambda_1+q_1)t}\right] \end{aligned} \tag{2}$$

   Either of the last two equations or with $u_0^* = \frac{q_0}{q_0+\lambda_0}$ and $u_1^* = \frac{q_1}{q_1+\lambda_1}$ plugged into either equations are accepted as correct answer. It should be noted that as $t \to \infty$, we have that $u(t) = u_1^*$ since $\left[1 - e^{-(\lambda_1+q_1)t}\right] \to 1$. When λ_1 and q_1 are larger, we will faster approach the new steady-state unemployment rate.

4. We can write equation (2) as

$$\begin{aligned} \frac{u(t)-u_0^*}{u_1^*-u_0^*} &= \left[1 - e^{-(\lambda_1+q_1)t}\right] \Leftrightarrow \\ \frac{1}{2} &= 1 - e^{-(\lambda_1+q_1)t} \Leftrightarrow \\ -(\lambda_1+q_1)t &= \ln(1) - \ln(2) \Leftrightarrow \\ t &= \frac{\ln(2)}{\lambda_1+q_1} \end{aligned}$$

   It is clear that the higher q_1 and λ_1, the faster we approach the new steady state unemployment. Conversely, if both are very low, the unemployment and employment stocks will change in a sluggish manner.

   The convergence rate slows down as we approach the new steady-state. To see this, we redo the calculations from above and ask how long time it takes to complete 95 percent of the way from u_0^* to u_1^*.

$$\begin{aligned} \frac{19}{20} &= 1 - e^{-(\lambda_1+q_1)t} \Leftrightarrow \\ -(\lambda_1+q_1)t &= \ln(1) - \ln(20) \Leftrightarrow \\ t &= \frac{\ln(20)}{\lambda_1+q_1} \end{aligned}$$

   Since $ln(20) - ln(2) = ln(10) \approx 2.3$, it is clear that the first 50 percent of the adjustment takes shorter time than the adjustment from 50 percent to 95 percent of the distance $u_1^* - u_0^*$.

5. The Mortensen-Pissarides (1994) model can give rise to the described unemployment dynamics with a lower outflow rate from unemployment as well as with a higher inflow rate to unemployment. This could happen in response to a higher productivity level at the production frontier, ε_u. The basic Diamond-Mortensen-Pissarides model cannot deliver endogenous changes to the job destruction rate following a productivity shock simply since the job destruction rate is exogenous. Answers answering the Diamond-Mortensen-Pissarides model with endogenous job destruction are obviously also correct.

6. Here, we consider the Mortensen-Pissarides (1994) model with an exogenous wage from the lecture on employment protection (but we abstract from firing costs). The endogenous variables are the unemployment rate, u, the labor market tightness, θ, and the reservation productivity, ε_d, and the equilibrium equations are given below. Notice that below, we use λ as the arrival rate of new productivity realizations and not as the exit rate from unemployment as in questions 1.1-1.4.

   Beveridge curve:
$$u = \frac{\lambda G(\varepsilon_d)}{\theta m(\theta) + \lambda G(\varepsilon_d)} \tag{3}$$

   Job creation equation:
$$\frac{h}{m(\theta)} = \frac{\varepsilon_u - \varepsilon_d}{r + \lambda} \tag{4}$$

   Job destruction equation:
$$\varepsilon_d = w - \frac{\lambda}{r + \lambda} \int_{\varepsilon_d}^{\varepsilon_u} (\varepsilon - \varepsilon_d)\, dG(\varepsilon) \tag{5}$$

   It is clear that the reservation productivity level, ε_d, is determined in the job destruction equation. Given ε_d, the labor market tightness, θ, is determined in the job creation equation. Given ε_d and θ, the unemployment rate, u is determined in the Beveridge equation.

   A higher productivity level at the production frontier, ε_u, decreases the reservation productivity level, ε_d. The reason is that the greater production potential of the match, due to the higher ε_u, implies that the firm is more reluctant to terminate a low productive match. Of course, this is partly due to the stochastic process of the match productivity being memoryless as it is a Poisson jump process.

   In the job creation equation, it is clear that a higher ε_u and a lower ε_d increase the r.h.s., which implies that the probability of filling a vacancy, $m(\theta)$, should decrease and since $m(\theta)$ is decreasing in θ, we must have that the labor market tightness θ increases. The reason is that with a higher maximum productivity and a lower reservation productivity level, the expected discounted profits are higher, so firms create more vacancies (we also notice that matches are created at the productivity frontier).

   Finally, with more vacancies, the arrival rate of job offers for unemployed increase. Furthermore, with a lower productivity threshold, ε_d, the job destruction rate is lower. From the Beveridge curve, it is clear that both these factors reduce the unemployment rate, u.

Question 2

1. The reservation wage, x satisfies $rV_e(x) = rV_u$. Therefore, we can write

$$\begin{aligned}
rV_e(x) &= rV_u \Leftrightarrow \\
x + q(V_u - V_e(x)) + vm(v) \int_x^{\bar{w}} (V_e(\tilde{w}) - V_e(x)) dH(\tilde{w}) &= z + vm(v) \int_x^{\bar{w}} (V_e(\tilde{w}) - V_u) dH(\tilde{w}) \Leftrightarrow \\
x + vm(v) \int_x^{\bar{w}} (V_e(\tilde{w}) - V_u) dH(\tilde{w}) &= z + vm(v) \int_x^{\bar{w}} (V_e(\tilde{w}) - V_u) dH(\tilde{w}) \Leftrightarrow \\
x &= z
\end{aligned}$$

   Since the search technology is the same as employed and unemployed (and there is no disutility of working), the worker just needs to be paid (marginally above) the flow income as unemployed to accept a job. No rational firm will create vacancies with wages below the workers' common reservation wage $x = z$, so the lowest wage offered in the equilibrium is z.

2. Free-entry implies that

$$\begin{aligned} r\Pi_v(w) &= -h + m(v)[u + (1-u)G(w)](\Pi_e(w) - \Pi_v(w)) \Rightarrow \\ \Pi_e(w) &= \frac{h}{m(v)[u + (1-u)G(w)]} \end{aligned} \tag{6}$$

and that

$$\begin{aligned} r\Pi_e(w) &= y - w + (q + vm(v)[1 - H(w)])(\Pi_v(w) - \Pi_e(w)) \Rightarrow \\ r\Pi_e(w) &= y - w + (q + vm(v)[1 - H(w)])\Pi_e(w) \Leftrightarrow \\ \Pi_e(w) &= \frac{y - w}{r + q + vm(v)[1 - H(w)]} \end{aligned} \tag{7}$$

Combining equations (6) and (7), we obtain

$$\frac{h}{m(v)[u + (1-u)G(w)]} = \frac{y - w}{r + q + vm(v)[1 - H(w)]} \tag{8}$$

This equation implies that the expected costs of having a vacancy (l.h.s.) equates the expected profits of having a filled job (r.h.s.) due to the free-entry condition, which competes the expected profits of a vacancy down to zero.

The l.h.s. is the expected costs of having a vacancy since h is the flow costs and $\frac{1}{m(v)[u+(1-u)G(w)]}$ is the expected duration of a vacancy. $m(v)$ is multiplied to $[u + (1-u)G(w)]$ since whereas all unemployed workers accept the job offer with a wage of w, only employed workers earning less than w will accept this job offer.

The r.h.s. is the expected profits of having a filled job paying the wage w. The numerator is the flow profits, $y - w$, and the denominator is the user costs of capital. Notice that firms, which pay a higher wage, have lower user cost of capital, due to lower chances of of loosing workers through on the job search.

3. In steady-state, the inflow to the mass $G(w)$ equals the outflow from this mass. Since workers only move up the wage ladder, inflow is only from unemployment, u. The arrival rate of job offers is $vm(v)$, but only a share of $H(w)$ of the job offers come from firms offering a wage below w. The outflow from the mass $G(w)$ consist of two types, outflow due to exogenous job destruction with the rate, q, and outflow due to on-the-job search to higher paying firms with the rate $vm(v)[1 - H(w)]$.

$$\begin{aligned} uvm(v)H(w) = (1-u)G(w)[q + vm(v)(1 - H(w))] &\Leftrightarrow \\ G(w) = \frac{u}{1-u}\frac{vm(v)H(w)}{q + vm(v)(1 - H(w))} &\Leftrightarrow \\ G(w) = \frac{q}{vm(v)}\frac{vm(v)H(w)}{q + vm(v)(1 - H(w))} &\Leftrightarrow \\ G(w) = \frac{qH(w)}{q + vm(v)(1 - H(w))} & \end{aligned} \tag{9}$$

We obtain the familar equilibrium wage distribution from the Burdett-Mortensen (1998) model besides that the usual job arrival rates are endogenized here, i.e. $\lambda_u = \lambda_e = vm(v)$. As usual, we have that $G(w)$ stochastically dominates $H(w)$ because of on the job search. The higher q is and the lower $vm(v))$ is, the closer are $H(w)$ and $G(w)$ since this means a lower expected number of job offers per employment spell.

4. Evaluating equation (8) in z, we obtain

$$\frac{h}{m(v)[u + (1-u)G(z)]} = \frac{y - z}{r + q + vm(v)[1 - H(z)]} \Leftrightarrow$$

$$
\begin{aligned}
\frac{h}{m(v)} &= \frac{(y-z)u}{r+q+vm(v)} \Leftrightarrow \\
\frac{h}{m(v)} &= \frac{(y-z)\frac{q}{q+vm(v)}}{r+q+vm(v)} \Leftrightarrow \\
\frac{h}{m(v)} &= \frac{q(y-z)}{(r+q+vm(v))(q+vm(v))}
\end{aligned} \tag{10}
$$

Increasing y implies that either $vm(v)$ must increase or $m(v)$ must decrease. In both cases, this implies that v increases. Since the unemployment is decreasing in $vm(v)$, a higher y decreases the unemployment rate.

Thus, for demographic group 1 the average productivity is higher, whereby the flow profits are higher, i.e. $y_1 - w > y_2 - w$, which implies higher expected profits for firms with a filled job with a worker from demograhic group 1. Therefore, more firms create vacancies on island 1, i.e. $v_1 > v_2$. This implies a higher job offer rate on island 1, i.e. $v_1 m(v_1) > v_2 m(v_2)$, such that the unemployment rate is lower on island 1, i.e. $u_1 < u_2$.

This implies that equally productive workers being part of different demographic groups are facing different unemployment rates because of statistical discrimination. Obviously, this result is different compared to the frictionless case, where the unemployment rate would be zero for both demographic groups.

5. We consider the effect of a marginal higher y on $H(w)$ by differentiating the wage offer distribution function, $H(w)$ with rerspect to y

$$
\begin{aligned}
\frac{\partial H(w)}{\partial y} &= \frac{\frac{\partial vm(v)}{\partial y} vm(v) - (q+vm(v))\frac{\partial vm(v)}{\partial y}}{(vm(v))^2}\left[1-\sqrt{\frac{y-w}{y-z}}\right] - \frac{1}{2}\frac{q+vm(v)}{vm(v)}\left(\frac{y-w}{y-z}\right)^{-\frac{1}{2}}\frac{(y-z)-(y-w)}{(y-z)^2} \\
&= \frac{-q}{(vm(v))^2}\left[1-\sqrt{\frac{y-w}{y-z}}\right]\frac{\partial vm(v)}{\partial v}\frac{\partial v}{\partial y} - \frac{1}{2}\frac{q+vm(v)}{vm(v)}\left(\frac{y-w}{y-z}\right)^{-\frac{1}{2}}\frac{w-z}{(y-z)^2}
\end{aligned} \tag{11}
$$

Both terms on the r.h.s. are negative. In particular, we notice that $\frac{\partial vm(v)}{\partial v} > 0$ and $\frac{\partial v}{\partial y} > 0$. The fact that $\frac{\partial H(w)}{\partial y} < 0$ implies that the distribution of wage offers for demographic group 1 stochastically dominates the distribution of wage offers for demographic group 2, i.e. $H_1(w) < H_2(w)$. The implication is that workers from demographic group 1, on average, receive higher wage offers. Hence, workers from demographic group 1 are, on average, more productive and, therefore, they, on average, are offered a higher wage.

6. It is intuitive that with a higher job arrival rate when employed and sampling from a better wage distribution, the equilibrium wage distribution for workers in demographic group 1 stochastically dominates the distribution function for workers in demographic group 2, i.e. $G_1(w) < G_2(w)$. This means that, on average, employed workers in demographic group 1 receive a higher wage in equilibrium. Thus, statistical discrimination in a model with matching frictions also lead to wage differences for workers of equal productivity, who just happen to belong to different demographic groups.

This result can also be shown by differentiating $G(w)$ with respect to y.

$$
\frac{\partial G(w)}{\partial y} = \frac{q(q+vm(v)(1-H(w)))\frac{\partial H(w)}{\partial y} - qH(w)\left((1-H(w))\frac{\partial vm(v)}{\partial y} - vm(v)\frac{\partial H(w)}{\partial y}\right)}{(q+vm(v)(1-H(w)))^2}
$$

As argued above, we find a negative effect of a higher y on $G(w)$. The first term is negative since $\frac{\partial H(w)}{\partial y} < 0$, whereas the second term is negative since $-\frac{\partial vm(v)}{\partial y} < 0$ and $\frac{\partial H(w)}{\partial y} < 0$.